Published by Institute of Physics Publishing for SISSA

Received: *July 15, 2005*
Revised: *September 8, 2005*
Accepted: *September 9, 2005*
Published: *September 19, 2005*

The ruled vertex and D-E degenerations

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Abstract: We compute all genus topological amplitudes on configurations of ruled surfaces obtained by resolving lines of D-E singularities in compact Calabi-Yau threefolds. We find that our results are in agreement with genus zero mirror symmetry calculations, which is further evidence for the validity of the ruled vertex formalism for degenerate torus actions.

Keywords: String Duality, Topological Strings.

JHEP09(2005)044

Contents

1. Introduction

A vertex formalism for topological amplitudes in the presence of degenerate torus actions has been recently developed in [8]. In the present note we develop this formalism for configurations of ruled surfaces obtained by resolving genus zero curves of D-E singularities in compact Calabi-Yau threefolds. These are nontoric exceptional divisors in the ambient threefold whose higher genus contribution to the partition function could not be determined directly in the **A**-model with the methods available so far. The computations will be performed for a specific elliptically and $K3$ fibered threefold with a line of D_4 singularities, which is the resolution of the degree 36 hypersurface in $\mathbb{P}^4_{[1,1,4,12,18]}$.

The formalism of [8] computes the topological partition function of local ruled surfaces defined by localization with respect to a torus action. More precisely, in order to deal with noncompactness issues for the moduli spaces of stable maps, we have to employ localization in order to define the theory, not just as a computational tool. In more technical terms, we are dealing with residual Gromov-Witten theory [5] of a ruled surface embedded in a Calabi-Yau threefold.

Although this theory is mathematically well defined, it is not a priori clear if the resulting vertex formalism has concrete physical applications. In this paper we will address this question in the context of II**A** compactification on Calabi-Yau threefolds with a line of D-E singularities. We will show that the construction of [8] yields exact results for topological string amplitudes associated to curve classes supported on the exceptional locus of the resolution. This is a subsector of the full topological partition function of the compact Calabi-Yau threefold.

We should also mention the implications of our results for heterotic-II**A** duality. We obtain exact results for topological string amplitudes with nontrivial degree along the base of the type II**A** $K3$ fibration in the limit in which the II**A** $K3$ is very large keeping the size of exceptional curves finite. Interpreted in the heterotic theory, our formulae represent exact nonperturbative corrections to the vector multiplet moduli space in a certain limit of the heterotic theory which will be described in section 4.

2. D-E degenerations of Calabi-Yau threefolds

Throughout this paper we will be mainly interested in exceptional configurations of ruled surfaces obtained by resolving genus zero curves of D_n, $n \geq 4$ and respectively E_n, $n = 6,7,8$ singularities in Calabi-Yau threefolds.

Although it is not hard to construct compact Calabi-Yau threefolds with such singularities, in this section we will construct a local model which captures the essential features of the resolution. It will become shortly clear that the topological partition function for curve classes supported on the resolution does not depend on the embedding of the exceptional locus in a compact threefold. Therefore we will obtain a universal contribution to the topological amplitudes associated to a particular type of degeneration.

The local models will be realized as hypersurfaces $\widehat{X}$ in noncompact toric varieties Z with toric presentation of the form

$$\begin{array}{ccccc} Z_1 & Z_2 & U & V & W \\ 1 & 1 & a & b & c \end{array}$$

with disallowed locus $\{Z_1 = Z_2 = 0\}$. Note that Z is isomorphic to the total space of the rank three bundle $\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)$ over $\mathbb{P}^1$. We record below the defining equations and the values of a, b, c for singularities in the D and E series

$$\begin{array}{rllll} \mathsf{D}_n\,,\ n \geq 4: & a = -2(n-1) & b = -2(n-1) & c = -4 & U^2 + V^2W = W^{n-1} \\ \mathsf{E}_6: & a = -12 & b = -8 & c = -6 & U^2 + V^3 + W^4 = 0 \\ \mathsf{E}_7: & a = -18 & b = -12 & c = -8 & U^2 + V^3 + VW^3 = 0 \\ \mathsf{E}_8: & a = -30 & b = -20 & c = -12 & U^2 + V^3 + W^5 = 0\,. \end{array} \tag{2.1}$$

All hypersurfaces described in (2.1) are singular along the rational curve

$$U = V = W = 0\,. \tag{2.2}$$

The resolution X of a singular hypersurface $\widehat{X}$ in the above series can be obtained by performing toric blow-ups of the ambient variety Z. One can easily check that X has the structure of a fibration in affine surfaces over $\mathbb{P}^1$ with typical fiber isomorphic to the canonical resolution of a D-E surface singularity. Since the singularity type does not jump along the curve (2.2), there will not be any special fibers. Therefore each fiber contains a chain of rational curves $f_0, \ldots, f_{n-1}$ which intersect according to the corresponding Dynkin diagram. Taking into account the fibration structure over $\mathbb{P}^1$, the curves $f_0, \ldots, f_{n-1}$ will generate a collection of ruled surfaces $S_0, \ldots, S_{n-1}$ in X in one to one correspondence with the nodes of the Dynkin diagram. These surfaces intersect along common sections as shown in figure 1.

We will fix the notation so that S_1 denotes the central component of the exceptional locus which meets three other components along disjoint sections. Since S_0 is an irreducible ruling, it follows that it must be isomorphic to $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Otherwise S_0 would not have three disjoint sections. The degree of the remaining surfaces can be determined inductively

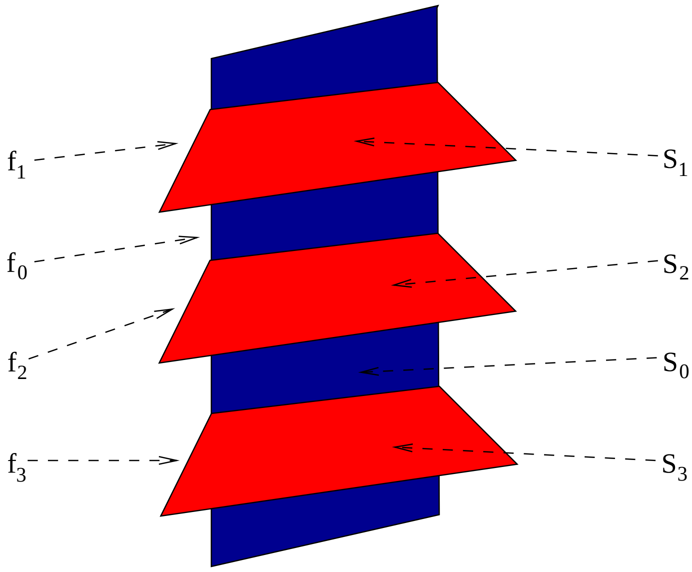

Figure 1: D_4 configuration of ruled surfaces.

starting from the central component. If two surfaces S_i, S_j meet along a common section C_{ij}, according to the adjunction formula we have

$$\deg(N_{S_i}C_{ij}) + \deg(N_{S_j}C_{ij}) = -2\,.$$

Using successively this formula we can determine the degrees e_i of the ruled surfaces S_i as shown in figure 1.

It is important to note for our purposes that the above configuration of ruled surfaces is not toric, but it admits a degenerate torus action.[1] The configuration is not toric because the central component $S_0 \simeq \mathbb{F}_0$ contains three marked sections C_{01}, C_{02}, C_{03} corresponding to the intersection loci with S_1, S_2, S_3. Therefore any torus action on X preserving the exceptional divisor $S_0 + \cdots + S_{n-1}$ must act with weight zero on the fiber f_0 of the central component S_0.

Let us consider a torus action with weights λ_{f_i}, $i = 0, \ldots, n-1$ on the fibers $f_0, \ldots, f_{n-1}$ of the rulings and $\lambda_{C_{ij}}$ on the sections C_{ij}. As noted above, $\lambda_{f_0} = 0$. In the following we will consider only torus actions satisfying the extra condition

$$\lambda_{C_{0i}} + \lambda_{f_i} = 0\,, \qquad i = 1, 2, 3\,.$$

This means that the fixed fibers f_0, f_0' are locally equivariant Calabi-Yau in the terminology of [5].

Given such a torus action one can employ the vertex formalism developed in [8] in order to compute the topological partition function. A priori this is a generating functional for residual Gromov-Witten invariants, which are defined [5] by summing over fixed loci in the

[1]Adopting the terminology introduced in [8], we will call a torus action on a Calabi-Yau threefold nondegenerate if the fixed locus is zero dimensional, and degenerate if the zero locus is higher dimensional.

moduli space of stable maps to X. In particular the invariants will depend on the choice of a torus action. However, in the present situation, one can check that the moduli space of stable maps $\overline{M}_{g,0}(X,\beta)$ is compact if β is a curve class supported on the the exceptional locus. Therefore the result has to be independent of the torus action, and moreover it has to agree with the Gromov-Witten invariants of any compact Calabi-Yau threefold which contains a degeneration of this form. We will check this statement by explicit computations in section 4.

Before explaining the construction of the topological partition function, let us briefly discuss the case of A_n degenerations. One might wonder why we chose to exclude this case from our considerations given the fact that these are the simplest degenerations from a geometric point of view. If we do not allow the singularity type to jump, an A_n degeneration will give rise to a toric configuration of ruled surfaces, and the topological partition function can be computed using the topological vertex formalism [1]. If we do allow the singularity type to jump, the resolution will contain chains of ruled surfaces with reducible fibers which were considered before in [8]. Typically, in such cases the residual Gromov-Witten invariants will depend on the choice of a torus action, therefore we would not obtain a universal contribution to the partition function of a generic Calabi-Yau threefold containing the degeneration in question. In conclusion the D and E series introduced above seem to be the most interesting testing ground for ideas discussed in this paper.

3. The topological partition function for D-E degenerations

In this section we explain the construction of the topological partition function for D and E degenerations. Since the central idea is the same in all examples it suffices to present one model in detail.

Let us start with the D_4 degeneration We have a central component $S_0 \simeq \mathbb{F}_0$ and three additional components $S_1, S_2, S_3 \simeq \mathbb{F}_2$. S_0 intersects each of the other components along a section C_{0i}, $i = 1,2,3$. Note that C_{0i} is the negative section on each S_i, $i = 1,2,3$ which squares to (-2). Taken in isolation, the partition function of each component can be easily written in the topological vertex formalism [1]. For the central component S_0 we have

$$Z_{S_0} = \sum_{R,R',Q,Q'} W_{RQ} W_{QR'} W_{R'Q'} W_{Q'R} q_{f_0}^{l(R)+l(R')} q_b^{l(Q)+l(Q')} \tag{3.1}$$

where $l(Q)$ is the total number of boxes of the Young diagram Q and W_{RQ} are functions of $q = e^{ig_s}$ defined as the large-N limit of the S-matrix of Chern-Simons theory

$$W_{RQ}(q) = \lim_{N\to\infty} q^{-\frac{N(l(R)+l(Q))}{2}} \frac{S_{RQ}(q,N)}{S_{00}(q,N)} .$$

Note that W_{PR} is symmetric in (P,R) and can be written in terms of Schur functions s_R:

$$W_{PR}(q) = s_R\left(q^{-i+1/2}\right) s_P\left(q^{R^i-i+1/2}\right)$$

where R^i is the length of the i-th row of Young diagram R and $i = 1,\ldots,\infty$. The variables q_{f_0}, q_b are the exponentiated Kähler parameters associated to the fiber class f_0 and respectively the section class $[C_{0i}]$, $i = 1, 2, 3$. The expression of the partition function for a local $\mathbb{F}_2$ surface is similar, except for some additional framing factors

$$Z_{S_i} = \sum_{S,S',P,P'} q^{-\kappa(P)} q^{\kappa(P')} W_{SP} W_{PS'} W_{P'Q'} W_{Q'P} q_{f_i}^{l(S)+l(S')+2l(P')} q_b^{l(P)+l(P')} \tag{3.2}$$

where $\kappa(R)$ is defined by the formula

$$\kappa(R) = 2 \sum_{\square \in R} (i(\square) - j(\square)) \tag{3.3}$$

in which $i(\square), j(\square)$ specify the position of a given box in the Young diagram. The variables q_{f_i}, $i = 1, 2, 3$ are the exponentiated Kähler parameter associated to the fiber classes f_i, $i = 1, 2, 3$.

In order to write down the partition function for the collection of surfaces $S_0 + \cdots + S_4$, we have to glue together the building blocks (3.1) and (3.2). At this point we have to use the ruled surface vertex found in [8] since the divisor $S_0 + \cdots + S_4$ is not toric. Let us briefly recall the construction of [8].

The topological partition function for a local ruled surface was obtained by a TQFT algorithm based on a decomposition of the ruled surface in basic building blocks. Given a ruled surface S over a curve Σ of genus g (possibly with reducible fibers), the decomposition in question is induced by a decomposition of the base Σ in pairs of pants and caps. The caps may be absent if S is an irreducible ruling, but they are required if S has reducible fibers. We will not need the caps in the following.

To a pair of pants Δ occurring in the decomposition of Σ we associate a piece of a ruled surface obtained be restricting the fibration $S \to \Sigma$ to Δ. Therefore we obtain a $\mathbb{P}^1$ bundle P over Δ which is is characterized by an integer level p. The case we will need below corresponds to $p = 0$. Note that there is a degenerate torus action on P fixing to sections of the ruling. The topological partition function associated to the level zero ruled vertex P is a formal expression $V_{RR'}$ which depends on two arbitrary Young diagrams R, R' associated to the fixed sections. According to [8], $V_{RR'}$ is given by the following expression

$$V_{RR'} = \left(\sum_Q W_{RQ} W_{QR'} q_r^{l(Q)} \right)^{-1} \tag{3.4}$$

where q_r is the Kähler parameter of the ruling.

We now return to the construction of the partition function of the D_4 degeneration. Note that if we excise the three marked sections C_{0i}, $i = 1, 2, 3$ from the central component $\mathbb{F}_0$, we obtain precisely a ruled vertex $P = \Delta \times \mathbb{P}^1$. Therefore the partition function will be obtained by gluing the contributions of the three components S_i, $i = 1, 2, 3$ to the ruled vertex (3.4). An informal way of describing the gluing is by virtually decomposing the exceptional locus into basic building blocks using noncompact branes as shown in figure 2.

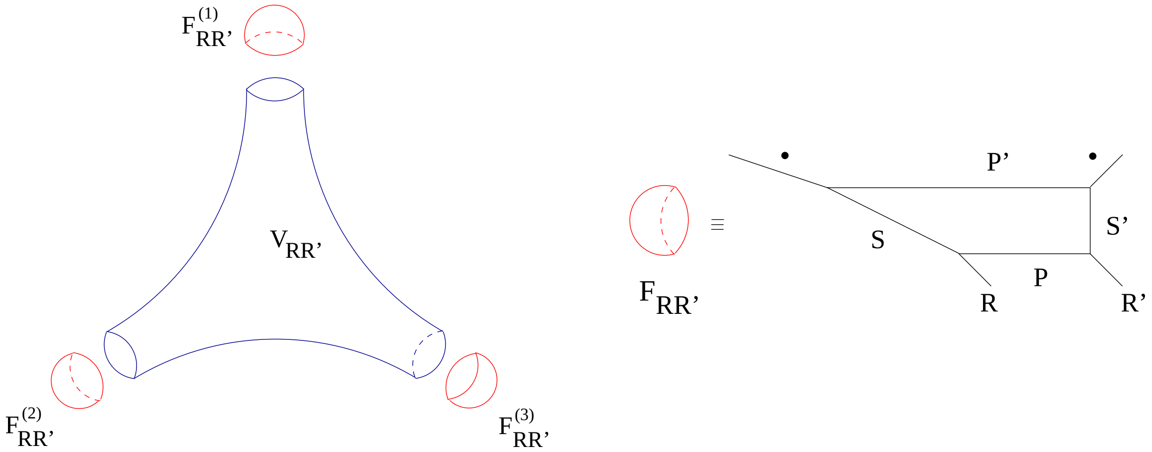

Figure 2: D_4 lego. The pair of pants represents the ruled vertex defined in [8]; the caps represent the contribution of the $\mathbb{F}_2$ surfaces.

This makes it clear that in the process of gluing we will have to use a topological vertex with all three legs nontrivial for each point of intersection between the sections C_{0i} and the sections f_0, f_0' fixed by the torus action. Collecting all the pieces we obtain the following expression

$$Z_{\mathsf{D}_4} = \sum_{R,R'} V_{R^tR'} q_{f_0}^{l(R)+l(R')} \prod_{i=1}^{3} F_{RR'}(q_{f_i}) \tag{3.5}$$

where $F(q_i)$ is the following formal expression

$$\begin{aligned} F_{RR'}(q_{f_i}) = \sum_{S,S',P,P'} & C_{RSP^t} C_{S^t\bullet(P')^t} C_{P'\bullet(S')^t} C_{S'R'P} q^{-\kappa(R)/2} q^{\kappa(S)/2} q^{-\kappa(S')/2} q^{-\kappa(P)/2} q^{\kappa(P')/2} \times \\ & \times q_{f_i}^{l(S)+l(S')+2l(P')} q_b^{l(P)+l(P')}. \end{aligned} \tag{3.6}$$

The partition function for other D_n as well as E_n degenerations can be easily written by analogy. The contribution of the central component $S_0 \simeq \mathbb{F}_0$ is the same in all cases. Therefore the generic form of the partition function is

$$\sum_{R,R'} V_{R^tR'} q_{f_0}^{l(R)+l(R')} \prod_{i=1}^{3} F^{(i)}_{RR'}(q_{f_i})$$

where the functions $F^{(i)}_{RR'}(q_{f_i})$ represent the contributions of the remaining three chains of ruled surfaces. Since these chains are toric, $F^{(i)}_{RR'}(q_{f_i})$ can be obtained using the topological vertex formalism as in [11].

For further reference, we will write down the first few terms in the expansion of the topological free energy for the D_4 singularity. Although the formulas (3.5), (3.6) are valid for arbitrary Kähler parameters, for the purpose of comparison with mirror symmetry calculations in a compact Calabi-Yau model, it suffices to set $q_{f_1} = q_{f_2} = q_{f_3} = q_f$. For notational simplicity we will also set $q_{f_0} = q_0$. Then up to degree two in q_b, q_0 and degree three in q_f, the free energy expansion reads

$$
\begin{aligned}
F_{\mathsf{D}_4} = & -\frac{1}{(2\sin\frac{g_s}{2})^2}(2q_b + 2q_0 + 6q_f + 4q_bq_0 + 6q_bq_f + 6q_0q_f) - \\
& -\frac{1}{(2\sin\frac{g_s}{2})^2}(6q_b^2q_0 + 6q_bq_0^2 + 18q_bq_0q_f + 12q_bq_f^2 + 6q_0q_f^2) - \\
& -\frac{1}{(2\sin\frac{g_s}{2})^2}(32q_b^2q_0^2 + 30q_b^2q_0q_f + 30q_bq_0^2q_f + 42q_bq_0q_f^2 + 18q_bq_f^3 + 2q_0q_f^3) + 9q_b^2q_0^2 - \\
& -\frac{1}{(2\sin\frac{g_s}{2})^2}(210q_b^2q_0^2q_f + 84q_b^2q_0q_f^2 + 84q_bq_0^2q_f^2 + 18q_b^2q_f^3 + 76q_bq_0q_f^3 + 2q_0^2q_f^3) + 48q_b^2q_0^2q_f - \\
& -\frac{1}{(2\sin\frac{g_s}{2})^2}(720q_b^2q_0^2q_f^2 + 222q_b^2q_0q_f^3 + 174q_bq_0^2q_f^3) + 147q_b^2q_0^2q_f^2 - \\
& -\frac{1850q_b^2q_0^2q_f^3}{(2\sin\frac{g_s}{2})^2} + 336q_b^2q_0^2q_f^3 + \text{multicover contributions} + \cdots .
\end{aligned}
\tag{3.7}
$$

One can also check that the multicover contributions have the expected BPS behavior. In the next section we will compare the genus zero terms in this expressions with the prepotential of an elliptically fibered compact Calabi-Yau model which contains generically a line of D_4 singularities.

4. A compact Calabi-Yau model with D_4 singularities

In order to test the ruled vertex formalism, we would like next to compare the expansion (3.7) with the topological free energy of a compact Calabi-Yau model which exhibits a D_4 degeneration as in figure 1. The most familiar examples are the elliptic fibrations over Hirzebruch surfaces $\mathbb{F}_n$ with lines of D or E singularities. As explained in [4], a generic D_4 singularity will correspond to G_2 enhanced symmetry with matter. To obtain the SO(8) configuration of surfaces, we have to take $n = 4$, which will correspond to an SO(8) theory without matter.

The vertices of the ∇ polytope are as follows:

$$\mathtt{V}_1 = (-1,0,2,3),\ \mathtt{V}_2 = (0,0,-1,0),\ \mathtt{V}_3 = (0,0,0,-1),\ \mathtt{V}_4 = (1,4,2,3),\ \mathtt{V}_5 = (0,-1,2,3).$$

The Calabi-Yau hypersurface X has $h^{1,1}(X) = 7$, $h^{2,1}(X) = 271$,[2] in agreement with the existence of an SO(8) in the Picard lattice. X can also be viewed as the resolution of the degree 36 hypersurface in $\mathbb{P}^4_{[1,1,4,12,18]}$. The other toric Calabi-Yau divisors are associated with the following points in the ∇ polytope:

$$\mathtt{B} = (0,0,2,3),\ \mathtt{D}_0 = (0,2,2,3),\ \mathtt{D}_1 = (0,1,1,1),\ \mathtt{D}_\infty = (0,1,2,3).$$

[2] We have used the program POLYHEDRON written by Philip Candelas.

The Mori cone of the Calabi-Yau X is obtained employing the method of the piecewise linear functions [18]:

$$\begin{array}{rrrrrrrrrr} & V_1 & V_2 & V_3 & V_4 & V_5 & B & D_0 & D_1 & D_\infty \\ e^{(1)}: & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 3 & 1 \\ e^{(2)}: & 0 & 0 & 1 & 0 & -2 & 3 & 0 & -2 & 0 \\ e^{(3)}: & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ e^{(4)}: & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \\ e^{(5)}: & 1 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 \end{array} \tag{4.1}$$

and is five dimensional; the same is true for its dual Kähler cone. The Kähler cone of the ambient toric variety Z is nine dimensional. If we denote by $\iota : X \hookrightarrow Z$ the inclusion map, we have the following exact sequence

$$0 \to \mathcal{K} \to H^2(Z) \xrightarrow{\iota^*} H^2(X) \to \mathcal{C} \to 0 \,.$$

The image of the map $\iota^* : H^2(Z) \to H^2(X)$ is five dimensional, and it has a four dimensional kernel $\mathcal{K}$ and two dimensional cokernel $\mathcal{C}$. Therefore, all of the exceptional surfaces S_1, S_2 and S_3 stay in the same divisor class in the toric Calabi-Yau hypersurface, which we identify with D_1. We also identify D_0 with the divisor class of the central component, D_∞ with the divisor class of the affine component and B with the section of the elliptic fibration.

Next, we perform a mirror symmetry computation to check the genus zero invariants obtained in the previous section. In order to do this, we need to identify the curve classes we are interested in with positive integer linear combinations of the generators of the Mori cone. We arrive at the following relations:

$$\begin{aligned} &f_0 = e^{(1)}\,, \qquad f_1 = f_2 = f_3 = f = e^{(2)} + e^{(3)} + 2e^{(4)}\,, \qquad f_\infty = e^{(3)}\,, \qquad C_0 = e^{(5)}, \\ &C_0^{\mathbb{F}_4} = 2e^{(3)} + e^{(5)}\,, \qquad f^{\mathbb{F}_4} = e^{(4)}, \end{aligned} \tag{4.2}$$

where $C_0^{\mathbb{F}_4}$ and $f^{\mathbb{F}_4}$ are the negative section and fiber classes of the base $\mathbb{F}_4$ respectively. We note that the exceptional fiber class fractionates in the ambient toric variety, and therefore, in order to compute the genus zero invariants in (3.7) we need to compute the prepotential up to order 16 in the instanton expansion. The generators of the Kähler cone are given by

$$\begin{aligned} &\mathcal{R}_1 = 12V_4 + 2B + 6D_0 + 3D_1 + 4D_\infty\,, \quad \mathcal{R}_2 = 18V_4 + 3B + 9D_0 + 4D_1 + 6D_\infty\,, \quad \mathcal{R}_5 = V_4\,, \\ &\mathcal{R}_3 = 26V_4 + 4B + 13D_0 + 6D_1 + 8D_\infty\,, \quad \mathcal{R}_4 = 40V_4 + 6B + 20D_0 + 9D_1 + 13D_\infty\,, \end{aligned}$$

and the Kähler form is $J = \sum_{i=1}^5 t_i \mathcal{R}_i$. The nontrivial triple intersections numbers are [13]

$$\begin{array}{lllll} D_0^3 = 8\,, & D_0^2 D_\infty = -2\,, & D_\infty^3 = 8\,, & D_0^2 D_1 = -6\,, & D_1^3 = 24\,, \\ BD_\infty^2 = 4\,, & B^2 D_\infty = 2\,, & B^3 = 8\,, & D_0^2 V_4 = -2\,, & D_0 D_\infty V_4 = 1\,, \\ D_\infty^2 V_4 = -2\,, & D_0 D_1 V_4 = 3\,, & D_1^2 V_4 = -6\,, & BD_\infty V_4 = 1\,, & B^2 V_4 = -2\,. \end{array}$$

The fundamental period for the mirror of X is given by

$$\begin{aligned}\varpi_0 &= \sum_{n_1,n_2,n_3,n_4,n_5} h_0 \\ &= \sum_{n_1,n_2,n_3,n_4,n_5} \frac{z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4} z_5^{n_5}}{\Gamma(1+n_1)\Gamma(1+n_2)\Gamma(1+n_5)^2\Gamma(1-2n_2+n_4)\Gamma(1+3n_2+n_3-2n_4)} \times \\ &\quad \times \frac{\Gamma(1+3n_1)}{\Gamma(1-2n_1+n_3-2n_5)\Gamma(1+3n_1-2n_2)\Gamma(1+n_1-2n_3+n_4)},\end{aligned}$$

where z_i, $i = 1, \ldots, 5$ are the large complex structure coordinates. We denote by ϖ_i, $i = 1, \ldots, 5$ the periods that behave asymptotically like $\ln z_i$, $i = 1, \ldots, 5$; these are given by

$$\varpi_i = \varpi_0 \ln z_i + g_i\,, \qquad i = 1, \ldots, 5\,, \tag{4.3}$$

where

$$\begin{aligned}g_1 &= \sum_{n_1,n_2,n_3,n_4,n_5} \Big[h_0(3S_{3n_1} - S_{n_1} + 2S_{-2n_1+n_3-2n_5} - 3S_{3n_1-2n_2} - S_{n_1-2n_3+n_4}) + \\ &\qquad + 2h_3 - 3h_4 - h_5\Big]\,, \\ g_2 &= \sum_{n_1,n_2,n_3,n_4,n_5} \Big[h_0(-S_{n_2} + 2S_{-2n_2+n_4} - 3S_{3n_2+n_3-2n_4} + 2S_{3n_1-2n_2}) + 2h_1 - 3h_2 + 2h_4\Big]\,, \\ g_3 &= \sum_{n_1,n_2,n_3,n_4,n_5} \Big[h_0(-S_{3n_2+n_3-2n_4} - S_{-2n_1+n_3-2n_5} + 2S_{n_1-2n_3+n_4}) - h_2 - h_3 + 2h_5\Big]\,, \\ g_4 &= \sum_{n_1,n_2,n_3,n_4,n_5} \Big[h_0(-S_{n_4} + 2S_{3n_2+n_3-2n_4} - S_{n_1-2n_3+n_4}) - h_1 + 2h_2 - h_5\Big]\,, \\ g_5 &= \sum_{n_1,n_2,n_3,n_4,n_5} \Big[h_0(-2S_{n_5} + 2S_{-2n_1+n_3-2n_5}) + 2h_3\Big]\,.\end{aligned}$$

In the above we have introduced the notation $S_n = \sum_{k=1}^{n} \frac{1}{k}$ and we have defined

$$\begin{aligned}h_1 &= \frac{z_1^{n_1} z_2^{n_2} z_3^{n_3} (-z_4)^{n_4} z_5^{n_5}}{\Gamma(1+n_1)\Gamma(1+n_2)\Gamma(1+n_5)^2\Gamma(1+3n_2+n_3-2n_4)\Gamma(1-2n_1+n_3-2n_5)} \times \\ &\quad \times \frac{\Gamma(1+3n_1)\Gamma(2n_2-n_4)}{\Gamma(1+3n_1-2n_2)\Gamma(1+n_1-2n_3+n_4)}\,, \\ h_2 &= \frac{z_1^{n_1} (-z_2)^{n_2} (-z_3)^{n_3} z_4^{n_4} z_5^{n_5}}{\Gamma(1+n_1)\Gamma(1+n_2)\Gamma(1+n_5)^2\Gamma(1-2n_2+n_4)\Gamma(1-2n_1+n_3-2n_5)} \times \\ &\quad \times \frac{\Gamma(1+3n_1)\Gamma(-3n_2-n_3+2n_4)}{\Gamma(1+3n_1-2n_2)\Gamma(1+n_1-2n_3+n_4)}\,, \\ h_3 &= \frac{z_1^{n_1} z_2^{n_2} (-z_3)^{n_3} z_4^{n_4} z_5^{n_5}}{\Gamma(1+n_1)\Gamma(1+n_2)\Gamma(1+n_5)^2\Gamma(1-2n_2+n_4)\Gamma(1+3n_2+n_3-2n_4)} \times \\ &\quad \times \frac{\Gamma(1+3n_1)\Gamma(2n_1-n_3+2n_5)}{\Gamma(1+3n_1-2n_2)\Gamma(1+n_1-2n_3+n_4)}\,, \\ h_4 &= \frac{(-z_1)^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4} z_5^{n_5}}{\Gamma(1+n_1)\Gamma(1+n_2)\Gamma(1+n_5)^2\Gamma(1-2n_2+n_4)\Gamma(1+3n_2+n_3-2n_4)} \times\end{aligned}$$

$$\times \frac{\Gamma(1+3n_1)\Gamma(-3n_1+2n_2)}{\Gamma(1-2n_1+n_3-2n_5)\Gamma(1+n_1-2n_3+n_4)}\,,$$

$$h_5 = \frac{(-z_1)^{n_1} z_2^{n_2} z_3^{n_3} (-z_4)^{n_4} z_5^{n_5}}{\Gamma(1+n_1)\Gamma(1+n_2)\Gamma(1+n_5)^2\Gamma(1-2n_2+n_4)\Gamma(1+3n_2+n_3-2n_4)} \times$$
$$\times \frac{\Gamma(1+3n_1)\Gamma(-n_1+2n_3-n_4)}{\Gamma(1-2n_1+n_3-2n_5)\Gamma(1+3n_1-2n_2)}\,.$$

Following [6, 10], we can now write down the mirror map

$$t_i = -\frac{\varpi_i}{\varpi_0} = -\ln z_i - \frac{g_i}{\varpi_0}\,, \qquad i = 1,\ldots,5\,. \tag{4.4}$$

In order to compute the genus zero Gromov-Witten invariants of X, we need to define the following series

$$\varpi_{ij} = \left(\frac{\partial^2 \Pi}{\partial\rho_i\partial\rho_j}\right)_{|\rho_i=\rho_j=0}\,, \qquad i,j = 1,\ldots,5\,, \tag{4.5}$$

where

$$\Pi = \sum_{n_1,n_2,n_3,n_4,n_5} \prod_{i=1}^{5} z_i^{n_i+\rho_i} \frac{\Gamma(1+3n_1+3\rho_1)}{\Gamma(1+3\rho_1)} \frac{\Gamma(1+\rho_1)}{\Gamma(1+n_1+\rho_1)} \frac{\Gamma(1+\rho_2)}{\Gamma(1+n_2+\rho_2)} \frac{\Gamma(1+\rho_5)^2}{\Gamma(1+n_5+\rho_5^2)} \times$$
$$\times \frac{\Gamma(1-2\rho_2+\rho_4)}{\Gamma(1-2n_2+n_4-2\rho_2+\rho_4)} \frac{\Gamma(1+3\rho_2+\rho_3-2\rho_4)}{\Gamma(1+3n_2+n_3-2n_4+3\rho_2+\rho_3-2\rho_4)} \times$$
$$\times \frac{\Gamma(1-2\rho_1+\rho_3-2\rho_5)}{\Gamma(1-2n_1+n_3-2n_5-2\rho_1+\rho_3-2\rho_5)} \frac{\Gamma(1+3\rho_1-2\rho_2)}{\Gamma(1+3n_1-2n_2+3\rho_1-2\rho_2)} \times$$
$$\times \frac{\Gamma(1+\rho_1-2\rho_3+\rho_4)}{\Gamma(1+n_1-2n_3+n_4+\rho_1-2\rho_3+\rho_4)}\,.$$

We will not explicitly write down here the series $\varpi_{ij}, i,j = 1,\ldots,5$. They are available upon request.

Taking into account the triple intersection numbers (4.2), a consequence of mirror symmetry is that the following equations hold true (see e.g. [10]):

$$\begin{aligned}
\partial_{t_1}\mathcal{F}_X^{(0)} = {} & -36\varpi_{11} - 108\varpi_{12} - 156\varpi_{13} - 240\varpi_{14} - 6\varpi_{15} - 81\varpi_{22} - 234\varpi_{23} - 360\varpi_{24} - \\
& -9\varpi_{25} - 169\varpi_{33} - 520\varpi_{34} - 13\varpi_{35} - 400\varpi_{44} - 20\varpi_{45}\,, \\
\partial_{t_2}\mathcal{F}_X^{(0)} = {} & -54\varpi_{11} - 162\varpi_{12} - 234\varpi_{13} - 360\varpi_{14} - 9\varpi_{15} - 120\varpi_{22} - 348\varpi_{23} - 534\varpi_{24} - \\
& -12\varpi_{25} - 252\varpi_{33} - 774\varpi_{34} - 18\varpi_{35} - 594\varpi_{44} - 27\varpi_{45}\,, \\
\partial_{t_3}\mathcal{F}_X^{(0)} = {} & -78\varpi_{11} - 234\varpi_{12} - 338\varpi_{13} - 520\varpi_{14} - 13\varpi_{15} - 174\varpi_{22} - 504\varpi_{23} - 774\varpi_{24} - \\
& -18\varpi_{25} - 364\varpi_{33} - 1120\varpi_{34} - 26\varpi_{35} - 860\varpi_{44} - 40\varpi_{45}\,, \\
\partial_{t_4}\mathcal{F}_X^{(0)} = {} & -120\varpi_{11} - 360\varpi_{12} - 520\varpi_{13} - 800\varpi_{14} - 20\varpi_{15} - 267\varpi_{22} - 774\varpi_{23} - 1188\varpi_{24} - \\
& -27\varpi_{25} - 560\varpi_{33} - 1720\varpi_{34} - 40\varpi_{35} - 1320\varpi_{44} - 60\varpi_{45}\,, \\
\partial_{t_5}\mathcal{F}_X^{(0)} = {} & -3\varpi_{11} - 9\varpi_{12} - 13\varpi_{13} - 20\varpi_{14} - 6\varpi_{22} - 18\varpi_{23} - 27\varpi_{24} - 13\varpi_{33} - 40\varpi_{34} - \\
& -30\varpi_{44}\,.
\end{aligned}$$

where $\mathcal{F}_X^{(0)}$ is the genus zero free energy of X. These equations completely determine the prepotential; after a long computation we obtain the instanton expansion presented below:

$$\begin{aligned}\mathcal{F}_X^{(0)} = & -2q_1 - 2q_5 - \frac{1}{4}q_1^2 - \frac{1}{4}q_5^2 - 4q_1q_5 - 6q_1^2q_5 - 6q_1q_5^2 - 6q_2q_3q_4^2 - \frac{65}{8}q_1^2q_5^2 - \\ & -6q_2q_3q_4^2(q_1+q_5) - 18q_1q_2q_3q_4^2q_5 - 30q_1^2q_2q_3q_4^2q_5 - 30q_1q_2q_3q_4^2q_5^2 - \frac{3}{4}q_2^2q_3^2q_4^4 - \\ & -210q_1^2q_2q_3q_4^2q_5^2 - 6q_1q_2^2q_3^2q_4^4 - 12q_2^2q_3^2q_4^4q_5 - \frac{3}{4}q_2^2q_3^2q_4^4(q_1^2+q_5^2) - 42q_1q_2^2q_3^2q_4^4q_5 - \\ & -84q_1^2q_2^2q_3^2q_4^4q_5 - 84q_1q_2^2q_3^2q_4^4q_5^2 - \frac{2889}{4}q_1^2q_2^2q_3^2q_4^4q_5^2 - \frac{2}{9}q_2^3q_3^3q_4^6 - 2q_1q_2^3q_3^3q_4^6 - \\ & -18q_2^3q_3^3q_4^6q_5 - 2q_1^2q_2^3q_3^3q_4^6 - 18q_2^3q_3^3q_4^6q_5^2 - 76q_1q_2^3q_3^3q_4^6q_5 - 174q_1^2q_2^3q_3^3q_4^6q_5 - \\ & -222q_1q_2^3q_3^3q_4^6q_5^2 - 1850q_1^2q_2^3q_3^3q_4^6q_5^2 + \cdots ,\end{aligned}$$

where $q_i = e^{-t_i}$, $i = 1, \ldots, 5$. Taking into account the relations

$$q_0 = e^{-t_1}, \qquad q_f = e^{-(t_2+t_3+2t_4)}, \qquad q_b = e^{-t_5}, \tag{4.6}$$

we find that the genus zero expansion of the compact model prepotential agrees with the expression (3.7). This is very convincing evidence for the ruled vertex formalism strengthening the enumerative tests performed in [8].

We would like to conclude this section with a brief discussion of the implications of our results for heterotic-type-II**A** duality. The compact model considered here has the interesting property of being related by string duality to two distinct heterotic theories [2]. One possible heterotic dual is the $E_8 \times E_8$ heterotic string compactified on $K3 \times T^2$ with instanton numbers $(8, 16)$ and unbroken gauge group $SO(8)$ corresponding to the line of D_4 singularities. Another heterotic dual is a $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string on $K3 \times T^2$ with instanton number 24 and unbroken $SO(8)$ gauge group. In both cases we have no charged multiplets.

In section 3 we have obtained exact results for topological amplitudes supported on the exceptional locus of the D_4 degeneration. This is a particular limit of the theory in which we are sending the size of the $K3$ fiber and the elliptic fiber to ∞ keeping the sizes of the base and the exceptional divisors fixed. Note that this is not the typical weak coupling heterotic limit which has been studied intensively in the string duality literature [12, 15, 3, 9, 17, 14]. In fact it is an easy exercise to understand the corresponding limit of the heterotic theory using the duality map explained in section 5 of [2]. For simplicity, we will set the B-field on the two-cycles of the II**A** Calabi-Yau space to zero. This corresponds to a square heterotic two-torus. Then the limit consists of taking one of the radii of the two torus to infinity keeping the other radius, as well as the $SO(8)$ Wilson lines fixed. We also keep the heterotic dilaton-axion $S = t_b$ fixed. In this limit the higher order terms in q_b in formula (3.7) can be interpreted as nonperturbative corrections to the vector multiplet moduli space depending on the Wilson line moduli. Similar results have been obtained before for generic Kähler parameters in [16] up to genus two on the base. Our formulas include all genus corrections on the base of the II**A** threefold, but they are valid only in the limit described above. In would be very interesting to further explore the consequences of our formulae for the heterotic string, especially in the strong coupling regime.

Acknowledgments

We would like to thank Eleonora Dell'Aquila for collaboration at an early stage of the project and Natalia Saulina for comments on the manuscript. D.-E. D. was partially supported by an Alfred P. Sloan fellowship and the work of B.F. was partially supported by DOE grant DE-FG02-96ER40949.

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